Concentration for independent random variables with heavy tails

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Abstract

If a random variable is not exponentially integrable, it is known that no concentration inequality holds for an infinite sequence of independent copies. Under mild conditions, we establish concentration inequalities for finite sequences of n independent copies, with good dependence in n.

1 Introduction

This paper continues the study of the concentration of measure phenomenon for product probability measures. A detailed account of this topic and its applications is given in [11]. Let us recall an important method for this problem: if μ (say on $\mathbb{R}^d$) satisfies a spectral gap (or Poincaré) inequality

$$\mathbf{Var}_\mu(f) \leq C \int |\nabla f|^2 d\mu, \quad \text{for all locally Lipschitz } f : \mathbb{R}^d \to \mathbb{R},$$

then Lipschitz functions are exponentially concentrated [8, 7]. More precisely every 1-Lipschitz function F in the Euclidean distance, with median m_F, satisfies $\mu(|F-m_F| > t) \leq 6\exp(-t/(2\sqrt{C}))$ for $t > 0$. Since the Poincaré inequality has the so-called tensorisation property, the same property holds for μ^n for all $n \geq 1$. Similarly, the logarithmic Sobolev inequality (see e.g. [10]) yields dimension free Gaussian concentration, whereas recent inequalities devised by Latała and Oleszkiewicz [9] provide intermediate rates, see also [4, 15, 3]. Note that these results only concern distributions with exponential or faster decay. This was explained by Talagrand [14]. Together with his famous result for products of exponential laws he observed the following: if μ is a probability measure on $\mathbb{R}$ such that there exist $h > 0$ and $\varepsilon_{1/2} > 0$ such that for all $n \geq 1$ and all $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq \frac{1}{2}$, one has

$$\mu^n(A + [-h, h]^n) \geq \frac{1}{2} + \varepsilon_{1/2}$$

then μ has exponential tails, that is there exist positive constants C_1, C_2 such that $\mu([x, +\infty)) \leq C_1 e^{-C_2 x}$, $x \in \mathbb{R}$. A similar property for all $p \in (0, 1)$ instead of just $p = 1/2$ implies that μ is the image of the symmetric exponential law by a map with finite modulus of continuity, as Bobkov and Houdré proved [5].

Thus when the tails of μ do not decay exponentially fast, there is no hope for dimension free concentration. This paper provides positive results in this case by investigating the size of enlargement h_n necessary to ensure a rise of the measure in dimension n. We study the more natural and also more difficult notion of Euclidean enlargement, and estimate h_n such that $\mu^n(A) \geq 1/2$ implies $\mu^n(A + h_n B_2^n) \geq \frac{1}{2} + \varepsilon$, where B_2^n is the Euclidean unit ball. By the above results we know that h_n has to tend to infinity as the dimension n increases. This question can be reformulated in terms of functions: we are looking for h_n such that for all n and all 1-Lipschitz functions $F : \mathbb{R}^n \to \mathbb{R}$ with median m_F, one has $\mu^n(F - m_F > h_n) \leq \frac{1}{2} - \varepsilon$.

We work in the setting of a Riemannian manifold (M, g) with a Borel probability measure which is absolutely continuous with respect to the volume measure. Our approach is based on the weak spectral gap inequality introduced by Röckner and Wang [12]. In this remarkable paper, these authors provide several necessary conditions for a measure to satisfy such a property, consequences for the corresponding semi-group and isoperimetric inequalities (see also [1, 16] for other developments). Our results complete and sharpen some of theirs. In Section 2 we give a characterization of measures on the real line with a weak spectral gap inequality. Section 3 shows that this functional inequality has a defective tensorisation property. We deduce isoperimetric and concentration inequalities for products in Sections 4 and 5. We illustrate our results with the examples of the power laws $\alpha(1 + |t|)^{-1-\alpha}dt/2$ for $\alpha > 0$ and the exponential type laws $\exp(-|t|^p)dt/(2\Gamma(1 + 1/p))$ for $p \in (0, 1)$. The latter should be of importance in the study of p-convex sets, as their analogues for $p \geq 1$ were in convex geometry (see e.g. [13]). We discuss our concentration consequences of the weak Poincaré inequality, in comparison with the ones of the recent article [16]. Our results are stronger, but the argument of Wang and Zhang can be improved in order to recover ours, and actually a slightly better though less explicit bound. The final section illustrates our method on a wide family of measures extending the laws $c_p \exp(-|t|^p)dt$, $p \in (0, 1)$.

Let μ be an absolutely continuous probability measure on a Riemannian manifold M. The modulus of gradient of a locally Lipschitz function $f : M \to \mathbb{R}$ can be defined as a whole by

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{d(x, y)}$$

where d is the geodesic distance. Following Röckner and Wang, we say μ satisfies a weak Poincaré inequality if there exists a function $\beta : (0, +\infty) \to \mathbb{R}^+$ such that every locally Lipschitz function $f : M \to \mathbb{R}$ satisfies for all $s > 0$ the inequality

$$\mathbf{Var}_\mu(f) \leq \beta(s) \int |\nabla f|^2 d\mu + s\, \mathbf{Osc}(f)^2.$$

Here $\mathbf{Osc}(f) = \sup f - \inf f$ is the total oscillation of the function f. The above mentioned authors used instead the quantity $\|f - \int f\, d\mu\|_\infty$. When this L_∞ essential supremum norm is with respect to the volume measure, the two quantities are the same up to a factor 2. We shall assume as we may, that β is non-increasing. Since $\mathbf{Var}_\mu(f) \le \mathbf{Osc}(f)^2/4$, the inequality is trivial when $s \ge 1/4$. In other words, one may set $\beta(s) = 0$ for $s \ge 1/4$. The real content of the inequality is when s is close to 0. If $\lim_{s\to 0} \beta(s) = b$, $b > 0$ then the measure satisfies a classical Poincaré or spectral gap inequality. Otherwise the speed of convergence to $+\infty$ is of great interest.

2 A measure-capacity criterion

This section provides an equivalent form of the weak Poincaré inequality, in terms of a comparison between capacity of sets and their measure. This point of view was put forward in [3] in order to give a natural unified presentation of the many functional inequalities appearing in the field. In dimension 1 this leads to a very effective necessary and sufficient condition for a measure to satisfy such an inequality, with a precise estimate of the function β. This completes the work by Röckner and Wang where several necessary conditions were provided.

In the following, $\mathbf{1}_S$ denotes the characteristic function of a set S, and $f_{|S}$ is the restriction of the function f to the set S. Given measurable sets $A \subset \Omega$, the capacity $\mathrm{Cap}_\mu(A, \Omega)$, is defined as

$$\begin{aligned}
\mathrm{Cap}_\mu(A,\Omega) &= \inf\left\{ \int |\nabla f|^2 d\mu;\ f_{|A} \ge 1,\ f_{|\Omega^c} = 0 \right\} \\
&= \inf\left\{ \int |\nabla f|^2 d\mu;\ \mathbf{1}_A \le f \le \mathbf{1}_\Omega \right\},
\end{aligned}$$

where the infimum is over locally Lipschitz functions. The latter equality follows from an easy truncation argument, reducing to functions with values in $[0, 1]$. Finally we defined in [4] the capacity of A with respect to μ when $\mu(A) < 1/2$ as

$$\mathrm{Cap}_\mu(A) := \inf\{\mathrm{Cap}(A,\Omega);\ A \subset \Omega,\ \mu(\Omega) \le 1/2\}.$$

Theorem 1. *Assume that for every $f : M \to \mathbb{R}$ and every $s \in (0, 1/4)$ one has*

$$\mathbf{Var}_\mu(f) \le \beta(s) \int |\nabla f|^2 d\mu + s\, \mathbf{Osc}(f)^2.$$

Then for every measurable $A \subset M$ with $\mu(A) < 1/2$, one has

$$\mathrm{Cap}_\mu(A) \ge \frac{\mu(A)}{4\beta(\mu(A)/4)}.$$

Proof. We start with assuming the weak Poincaré inequality. Let $A \subset \Omega$, where $\mu(\Omega) \le 1/2$. Let f be a locally Lipschitz function satisfying $\mathbf{1}_A \le f \le \mathbf{1}_\Omega$. By

Cauchy-Schwarz inequality,

$$\left(\int f\,d\mu\right)^2 = \left(\int f\mathbf{1}_\Omega\,d\mu\right)^2 \leq \mu(\Omega)\int f^2 d\mu.$$

Therefore $\mathbf{Var}_\mu(f) \geq \mu(\Omega^c)\int f^2 d\mu \geq \int f^2 d\mu/2$. Since the oscillation of f is at most 1, the weak Poincaré inequality yields for $s \in (0,1/4)$

$$\frac{1}{2}\mu(A) \leq \frac{1}{2}\int f^2 d\mu \leq \beta(s)\int |\nabla f|^2 d\mu + s.$$

This is valid for arbitrary f with $\mathbf{1}_A \leq f \leq \mathbf{1}_\Omega$. Hence we get

$$\frac{1}{2}\mu(A) \leq \beta(s)\,\mathrm{Cap}_\mu(A,\Omega) + s.$$

Taking the infimum over sets Ω with measure at most $1/2$ and containing A, we obtain for any $s \in (0,1/4)$

$$\frac{1}{\beta(s)}\left(\frac{\mu(A)}{2} - s\right)_+ \leq \mathrm{Cap}_\mu(A).$$

Note that as a function of $\mu(A)$ the above lower bound vanishes before $2s$ and then increases with slope $1/(2\beta(s))$. Taking supremum over s yields general lower bounds of the capacity by convex functions of the measure, vanishing at 0. More precisely we arrived at $\mathrm{Cap}_\mu(A) \geq \tilde{\beta}(\mu(A))$, where for $a \in (0,1/2)$,

$$\tilde{\beta}(a) = \sup_{s\in(0,1/4)}\left(\frac{\frac{a}{2}-s}{\beta(s)}\right)_+ = \sup_{s\in(0,a/2)}\frac{\frac{a}{2}-s}{\beta(s)}.$$

Note that

$$\frac{a}{4\beta(a/4)} \leq \tilde{\beta}(a) \leq \frac{a}{2\beta(a/2)},$$

where the lower bound corresponds to the choice $s = a/4$ and the upper bound relies on the non-increasing property of β. When this function satisfies a doubling condition ($\beta(2x) \geq c\beta(x)$) then the above bounds are the same up to a multiplicative constant. $\square$

Theorem 2. *Assume that γ is a non-increasing positive function on $(0,1/2)$. If every measurable $A \subset M$ with $\mu(A) \leq 1/2$ verifies*

$$\mathrm{Cap}_\mu(A) \geq \frac{\mu(A)}{\gamma(\mu(A))},$$

then for every locally Lipschitz function f and every $s \in (0,1/4)$ one has

$$\mathbf{Var}_\mu(f) \leq 12\gamma(s)\int |\nabla f|^2 d\mu + s\,\mathbf{Osc}(f)^2.$$

Proof. Fix $s \le 1/4$. Let m be a median of f under μ. Denote $\Omega_+ = \{f > m\}$ and $\Omega_- = \{f < m\}$. Then

$$\mathbf{Var}_\mu(f) \le \int (f-m)^2 d\mu = \int_{\Omega_+} (f-m)^2 d\mu + \int_{\Omega_-} (f-m)^2 d\mu.$$

We work separately on each of the latter two integrals. Consider $g = (f-m)_+$ as a function defined on Ω_+. Let $c = \inf\{t \ge 0;\ \mu(g^2 > t) \le s\}$. If $c = 0$ then $\mu(g > 0) \le s$ and $\int_{\Omega_+} g^2 d\mu \le s \max g^2$ and we are done for this half of space. Otherwise $\mu(g^2 > c) \le s$ and $\mu(g^2 \ge c) \ge s$. By our structural hypothesis of a Riemannian manifold with an absolutely continuous measure we can find a set Ω_0 with $\{g^2 > c\} \subset \Omega_0 \subset \{g^2 \ge c\}$ and $\mu(\Omega_0) = s$. Let $\rho > 1$. For $k < 0$ and integer, define $\Omega_k = \{g^2 \ge c\rho^k\}$. Then

$$\begin{aligned}
\int_{\Omega_+} g^2 d\mu &= \int_{\Omega_0} g^2 d\mu + \sum_{k<0} \int_{\Omega_k \setminus \Omega_{k+1}} g^2 d\mu \\
&\le s \sup(f-m)_+^2 + \sum_{k<0} c\rho^{k+1} \Big(\mu(\Omega_k) - \mu(\Omega_{k+1})\Big)
\end{aligned}$$

The second term is dealt with by Abel summation:

$$\sum_{k<0} \rho^{k+1} \Big(\mu(\Omega_k) - \mu(\Omega_{k+1})\Big) = (\rho - 1) \sum_{k<0} \rho^k \Big(\mu(\Omega_k) - \mu(\Omega_0)\Big)$$

Hence,

$$\int_{\Omega_+} g^2 d\mu \le s \sup(f-m)_+^2 + \sum_{k<0} c(\rho-1)\rho^k (\mu(\Omega_k) - s).$$

In order to use our hypothesis, note that it implies that for every A with measure at most $1/2$, one has $\mathrm{Cap}_\mu(A) \ge (\mu(A) - s)/\gamma(s)$. Indeed this is obvious if $s \ge \mu(A)$, whereas if $s \le \mu(A)$, $\mathrm{Cap}_\mu(A) \ge \mu(A)/\gamma(\mu(A)) \ge (\mu(A) - s)/\gamma(s)$ by the monotonicity of γ. Thus choosing

$$g_k = \min\left(1, \left(\frac{g - \sqrt{c\rho^{k-1}}}{\sqrt{c\rho^k} - \sqrt{c\rho^{k-1}}}\right)_+\right),$$

we have

$$\begin{aligned}
\mu(\Omega_k) - s &\le \gamma(s)\mathrm{Cap}_\mu(\Omega_k) \le \gamma(s) \int |\nabla g_k|^2 d\mu \\
&\le \gamma(s) \int_{\Omega_{k-1} \setminus \Omega_k} \frac{|\nabla g|^2}{c\rho^{k-1}(\sqrt{\rho} - 1)^2} d\mu.
\end{aligned}$$

Summing upon $k < 0$ we obtain

$$\begin{aligned}
\int_{\Omega_+} g^2 d\mu &\le s \sup(f-m)_+^2 + \gamma(s) \frac{\rho(\rho-1)}{(\sqrt{\rho}-1)^2} \sum_{k<0} \int_{\Omega_{k-1} \setminus \Omega_k} |\nabla g|^2 d\mu \\
&\le \gamma(s)\rho \frac{\sqrt{\rho}+1}{\sqrt{\rho}-1} \int_{\Omega_+} |\nabla f|^2 d\mu + s \sup(f-m)_+^2.
\end{aligned}$$

Summing up with a similar estimate for Ω_- and optimizing on ρ gives a slightly better estimate than the claimed one. $\square$

Theorem 3. *Let μ be a probability measure on $\mathbb{R}$. Assume that it is absolutely continuous with respect to Lebesgue measure and denote by ρ_μ its density. Let m be a median of μ. Let $\beta : (0, 1/2) \to \mathbb{R}^+$ be non-increasing. Let C be the optimal constant such that for all $f : \mathbb{R} \to \mathbb{R}$ and $s \in (0, 1/4)$,*

$$\mathbf{Var}_\mu(f) \leq C\beta(s) \int |\nabla f|^2 d\mu + s\, \mathbf{Osc}(f)^2.$$

Then $\frac{1}{4}\max(b_-, b_+) \leq C \leq 12 \max(B_-, B_+)$, where

$$\begin{aligned}
b_+ &= \sup_{x>m} \mu([x, +\infty)) \frac{1}{\beta(\mu([x, +\infty))/4)} \int_m^x \frac{1}{\rho_\mu} \\
b_- &= \sup_{x<m} \mu((-\infty, x]) \frac{1}{\beta(\mu((-\infty, x])/4)} \int_x^m \frac{1}{\rho_\mu} \\
B_+ &= \sup_{x>m} \mu([x, +\infty)) \frac{1}{\beta(\mu([x, +\infty)))} \int_m^x \frac{1}{\rho_\mu} \\
B_- &= \sup_{x<m} \mu((-\infty, x]) \frac{1}{\beta(\mu((-\infty, x]))} \int_x^m \frac{1}{\rho_\mu}.
\end{aligned}$$

Proof. We start with the lower bound on C. We have seen that the weak spectral gap inequality ensures that for all Ω with $\mu(A) \leq 1/2$ and $A \subset \Omega$, one has $\mathrm{Cap}_\mu(A, \Omega) \geq \mu(A)/(4C\beta(\mu(A)/4))$. Let $x > m$ and apply this inequality with $A = [x, +\infty)$ and $\Omega = (m, +\infty)$. It is easy to check that $\mathrm{Cap}_\mu([x, +\infty), (m, +\infty)) = 1/\int_m^x 1/\rho_\mu$. This yields $C \geq b_+/4$. A similar argument on the other side of the median m also gives $C \geq b_-/4$.

For the upper bound, we follow the argument of the proof of Theorem 2 with some modification. We start with writing that

$$\mathbf{Var}_\mu(f) \leq \int_m^{+\infty} |f - f(m)|^2 d\mu + \int_{-\infty}^m |f - f(m)|^2 d\mu.$$

We work separately on the right and on the left of m. We explain only for the right side; the left one is similar. To proceed the argument in the same way we need to check that any $A \subset (m, +\infty)$ verifies

$$\mathrm{Cap}_\mu(A, (m, +\infty)) \geq \frac{\mu(A)}{B_+\, \beta(\mu(A))}.$$

By hypothesis the above inequality holds when $A = [x, +\infty)$. It follows that it is valid for general A. Indeed, for any $A \subset (m, +\infty)$ one has $\mathrm{Cap}_\mu(A, (m, +\infty)) = \mathrm{Cap}_\mu([\inf A, +\infty), (m, +\infty))$. Since $\mu(A) \leq \mu([\inf A, +\infty))$ and $t \mapsto t/\beta(t)$ is non-decreasing the above inequality for half-lines implies it for general sets. $\square$

Corollary 4. *Let $d\mu(x) = e^{-\Phi(x)}dx$, $x \in \mathbb{R}$ be a probability measure. Let $\varepsilon \in (0,1)$. Assume that there exists an interval $I = (x_0, x_1)$ containing a median m of μ such that $|\Phi|$ is bounded on I, and Φ is twice differentiable outside I with*

$$\Phi'(x) \neq 0 \quad \text{and} \quad \frac{|\Phi''(x)|}{\Phi'(x)^2} \leq 1 - \varepsilon, \qquad x \notin I.$$

Let β be a decreasing function on $(0, 1/2)$. Assume that there exists $c > 0$ such that for all $x \notin I$ one has

$$\beta\left(\frac{e^{-\Phi(x)}}{\varepsilon|\Phi'(x)|}\right) \geq \frac{c}{\Phi'(x)^2}.$$

Then μ satisfies a weak Poincaré inequality with function $C\beta$ for some constant $C > 0$.

Proof. We evaluate the quantity B_+ in the above theorem. The study of B_- is similar. For $x \geq x_1$, we have

$$\left(\frac{e^{\Phi}}{\Phi'}\right)'(x) = e^{\Phi(x)}\left(1 - \frac{\Phi''(x)}{\Phi'(x)^2}\right) \geq \varepsilon e^{\Phi(x)}.$$

Therefore by integration

$$\int_m^x e^{\Phi} \leq \int_m^{x_1} e^{\Phi} + \int_{x_1}^x e^{\Phi} \leq (x_1 - m)e^M + \frac{1}{\varepsilon}\left(\frac{e^{\Phi(x)}}{\Phi'(x)} - \frac{e^{\Phi(x_1)}}{\Phi'(x_1)}\right),$$

where $M = \sup\{|\Phi(x)|;\ x \in I\}$. Similar calculations give

$$(2-\varepsilon)e^{-\Phi(x)} \geq \left(-\frac{e^{-\Phi}}{\Phi'}\right)'(x) \geq \varepsilon e^{-\Phi(x)}.$$

Note that $\lim_{+\infty} e^{-\Phi}/\Phi' = 0$. Indeed this quantity is positive, since Φ' cannot change sign, and decreasing by the above bound. The limit has to be zero otherwise $e^{-\Phi(x)}$ would behave as c/x and would not be integrable. We obtain by integration for $x \geq x_1$,

$$\mu([x, +\infty)) \leq \frac{e^{-\Phi(x)}}{\varepsilon\Phi'(x)} \leq \frac{2-\varepsilon}{\varepsilon}\mu([x, +\infty)).$$

Combining these bounds on $\int_m^x e^{\Phi}$ and $\mu([x, +\infty))$ it is not hard to show that B_+ is finite. ☐

Example 1. For $\alpha > 0$, the measure $dm_\alpha(t) = \alpha(1+|t|)^{-1-\alpha}dt/2$, $t \in \mathbb{R}$ satisfies the weak spectral gap inequality with $\beta(s) = c_\alpha s^{-2/\alpha}$. This was proved differently in [12], our next result improve on theirs.

Example 2. For $p \in (0,1)$, the measure $d\nu_p(t) = e^{-|t|^p}/(2\Gamma(1+1/p))$, $t \in \mathbb{R}$ satisfies the inequality with $\beta(s) = d_p \log(2/s)^{\frac{2}{p}-2}$.

Remark 3. In the above examples, the functions β are best possible up to a multiplicative constant (we could write an analogue of the previous corollary, providing a necessary condition for a weak Poincaré inequality to hold with β, with a similar proof). Since these functions β satisfy the doubling condition, our theorem describes all real measures enjoying the same functional inequality.

3 Tensorisation

It is classical that the Poincaré inequality enjoys the tensorisation property. When β has infinite limit at 0, the weak spectral gap inequality does not tensorise. We shall give geometric evidence for this in the section related to isoperimetry. However if μ satisfies the inequality with a function β, then μ^n satisfies a weak spectral gap inequality with a worse function.

Theorem 5. *Assume that for every $f : M \to \mathbb{R}$ and every $s \in (0, 1/4)$ one has*

$$\mathbf{Var}_\mu(f) \leq \beta(s) \int |\nabla f|^2 d\mu + s\,\mathbf{Osc}(f)^2.$$

Let $n \geq 1$. Then for every $f : M^n \to \mathbb{R}$ and every $s \in (0, 1/4)$ one has

$$\mathbf{Var}_{\mu^n}(f) \leq \beta\Big(\frac{s}{n}\Big) \int |\nabla f|^2 d\mu^n + s\,\mathbf{Osc}(f)^2.$$

Proof. By the sub-additivity property of the variance,

$$\mathbf{Var}_{\mu^n}(f) \leq \sum_{i=1}^n \int \mathbf{Var}_\mu\Big(y_i \mapsto f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)\Big) \prod_{j \neq i} d\mu(x_j).$$

For each i the inner variance is at most

$$\beta(s) \int |\nabla_i f|^2(x_1, \ldots, y_i, \ldots, x_n) d\mu(y_i) + s\,\mathbf{Osc}\Big(y_i \mapsto f(x_1, \ldots, y_i, \ldots, x_n)\Big)^2.$$

The latter oscillation is less than or equal to $\mathbf{Osc}(f)$. Summing up we arrive at

$$\mathbf{Var}_{\mu^n}(f) \leq \beta(s) \int |\nabla f|^2 d\mu^n + ns\,\mathbf{Osc}(f)^2,$$

for all $s \in (0, 1/4)$. □

4 Isoperimetric inequalities

For $h > 0$ we denote the h-enlargement of a set $A \subset M$ in the geodesic distance by A_h. The boundary measure in the sense of μ is by definition

$$\mu_s(\partial A) = \liminf_{h \to 0} \frac{\mu(A_h \setminus A)}{h}.$$

The isoperimetric function encodes the minimal boundary measure of sets of prescribed measures:

$$I_\mu(a) = \inf\{\mu_s(\partial A);\ \mu(A) = a\}, \qquad a \in [0,1].$$

It was shown by Röckner and Wang that in the diffusion case, a weak spectral gap inequality for μ implies an isoperimetric inequality. We state here a consequence of their results.

Theorem 6 ([12]). *Let μ be a probability measure on (M,g), with density e^{-V} with respect to the volume measure. Assume that V is C^2 and such that $Ricci + \nabla\nabla V \geq Rg$ for some $R \leq 0$. If μ satisfies a weak spectral gap inequality with function β, with $\beta(1/8) \geq \varepsilon > 0$, then for every measurable $A \subset M$,*

$$\mu_s(\partial A) \geq c(\varepsilon, R)\frac{p}{\beta(p/2)},$$

where $p = \mu(A)(1-\mu(A)) \geq \min(\mu(A), \mu(A^c))/2$.

Remark 4. Comparing with a result of Röckner and Wang, showing that an isoperimetric inequality implies a weak spectral gap inequality, one notices that $\sqrt{\beta}$ is expected in the denominator (in the method, this loss comes from the necessity to estimate the underlying semi-group for large time instead of small time).

Corollary 7. *Under the hypothesis of the above theorem, the following isoperimetric inequality holds for all $n \geq 1$. For all $A \subset M^n$, one has*

$$\mu_s^n(\partial A) \geq c(\varepsilon, R)\frac{p}{\beta(p/(2n))},$$

where $p = \mu^n(A)(1-\mu^n(A))$.

Proof. The tensorisation result of the previous section provides a weak spectral gap inequality for μ^n with function $\beta(s/n)$. The latter theorem then applies. Note that the differential hypothesis on the density of μ remains valid for μ^n. We also used $\beta(1/(8n)) \geq \beta(1/8) \geq \varepsilon$. □

In the non-trivial cases when $\lim_0 \beta = +\infty$ the above lower bound of I_{μ^n} tends to zero as n increases. This has to be, as the following consideration of product sets shows. We shall assume that $I_\mu(t) = I_\mu(1-t)$ for all t (this is very natural, since regular sets have the same boundary measure as their complement). First note that for all $n \geq 1$, $h > 0$ and $A \subset M$ one has $(A^n)_h \subset (A_h)^n$, where $A^n \subset M^n$ is the cartesian product of n copies of A. Combining this with the definition of the boundary measure yields

$$\mu_s^n(\partial(A^n)) \leq n\mu(A)^{n-1}\mu_s(\partial A).$$

Taking infimum on A with prescribed measure, we get $I_{\mu^n}(a^n) \leq na^{n-1}I_\mu(a)$ for all $a \in (0,1)$. Thus for any fixed $t \in (0,1)$ one has when $n \geq \log(1/t)/\log(2)$

$$\begin{aligned} I_{\mu^n}(t) &\leq nt^{1-\frac{1}{n}} I_\mu(t^{\frac{1}{n}}) \leq 2nt I_\mu(1-t^{\frac{1}{n}}) \\ &= 2nt\, I_\mu\left(\frac{\log(1/t)}{n}(1+\varepsilon_t(n))\right) \\ &= 2t\log\left(\frac{1}{t}\right)\Theta\left(\frac{\log(1/t)}{n}(1+\varepsilon_t(n))\right)(1+\varepsilon_t(n)), \end{aligned}$$

where $\lim_n \varepsilon_t(n) = 0$ and $I_\mu(u) = u\Theta(u)$. If Θ tends to zero at zero then $\lim_n I_{\mu^n}(t) = 0$ with corresponding speed.

For even measures on $\mathbb{R}$ with positive density on a segment, Bobkov and Houdré [6, Corollary 13.10] proved that solutions to the isoperimetric problem can be found among half-lines, symmetric segments and their complements. More precisely, if ρ_μ is the density and R_μ the distribution function of μ, then denoting $J_\mu = \rho_\mu \circ R_\mu^{-1}$, one has for $t \in (0,1)$

$$I_\mu(t) = \min\left(J_\mu(t), 2J_\mu\Big(\frac{\min(t,1-t)}{2}\Big)\right).$$

This readily applies to our previous examples.

Example 5. For the measures $dm_\alpha(t) = \alpha(1+|t|)^{-1-\alpha}/2$ one gets $J_{m_\alpha}(t) = \alpha 2^{1/\alpha}\min(t,1-t)^{1+1/\alpha}$, and thus for $t \in (0,1/2)$,

$$I_{m_\alpha}(t) = \alpha t^{1+1/\alpha}.$$

The results of this section do not apply to m_α for lack of regularity. However for an even unimodal smoothed perturbation $\tilde{m}_\alpha$, up to a numerical constant, the same isoperimetric and weak spectral gap inequality hold. So there are constants such that for $t \leq 1/2$ and $n \geq \log(1/t)/\log 2$ one has

$$c_1(\alpha)\, t\left(\frac{t}{n}\right)^{2/\alpha} \leq I_{\tilde{m}_\alpha^n}(t) \leq c_2(\alpha)\, t\frac{\log(1/t)^{1+1/\alpha}}{n^{1/\alpha}}.$$

Example 6. For $p \in (0,1)$, and $d\mu_p(t) = \exp(-|t|^p)/(2\Gamma(1+1/p))$ similar estimates can be done. For $t \leq 1/2$, $I_{\nu_p}(t)$ is comparable to $t(\log(1/t))^{1-1/p}$. So for a suitable smoothed version of this measure, one gets

$$d_1(p)\, t\left(\log\left(\frac{n}{t}\right)\right)^{2(1-1/p)} \leq I_{\tilde{\nu}_p^n}(t) \leq d_2(p)\, t\log(1/t)\left(\log\left(\frac{n}{\log(1/t)}\right)\right)^{1-1/p},$$

which guarantees a convergence to zero with logarithmic speed in the dimension.

5 Concentration of measure

In this section, we shall derive concentration inequalities, that is lower bounds on the measure of enlargements of rather large sets, or equivalently deviation inequalities for Lipschitz functions. They can be approached via isoperimetric inequalities, which quantify the measure of infinitesimal enlargements. In our setting, we have seen in the previous section that the available methods provide loose isoperimetric bounds. Hence we come back to simpler and more robust techniques. It is known, since Gromov and Milman [8], that a Poincaré inequality yields exponential concentration. See e.g. [10] for subsequent developments. We show how a weak spectral gap inequality can be used to derive deviation inequalities for Lipschitz functions. Among the various available methods used for Poincaré inequalities, the one in Aida, Masuda and Shigekawa [2] is the most adapted.

Theorem 8. *Let μ satisfy a weak spectral gap inequality with function β. Let $F : M \to \mathbb{R}$ be a L-Lipschitz function with median m. Then for $k \geq 1$ and $s \in (0, 1/4)$, one has*

$$\mu(F - m > k) \leq \frac{s}{1 + L^2\beta(s)} + \mu(F - m > k - 1)\left(1 - \frac{1}{2(1 + L^2\beta(s))}\right). \quad (1)$$

Consequently

$$\mu(F - m > k) \leq 2s + \frac{\sqrt{e}}{2} \exp\Big(\frac{-k}{4L\sqrt{\beta(s)}}\Big). \quad (2)$$

Thus $\mu(|F - m| > k) \leq 6\,\Theta(k/L)$, where

$$\Theta(u) = \inf\Big\{s \in (0, 1/4];\ \exp\Big(\frac{-u}{4\sqrt{\beta(s)}}\Big) \leq s\Big\}$$

tends to 0 when u tends to infinity.

Proof. For notational convenience assume that $m = 0$. Let $\varepsilon > 0$. Let $\Phi : \mathbb{R} \to \mathbb{R}^+$ be a non-decreasing smooth function with $\Phi_{|(-\infty,\varepsilon]} = 0$, $\Phi_{|[1-\varepsilon,+\infty)} = 1$ and $\|\Phi'\|_\infty \leq 1 + 3\varepsilon$. Set $\Phi_k(t) = \Phi(t - k + 1)$. We apply the weak Poincaré inequality to $\Phi_k(F)$. Since $\mathbf{1}_{(k-1,+\infty)} \geq \Phi_k \geq \mathbf{1}_{[k,+\infty)}$ one has

$$\int \Phi_k(F)^2 d\mu \geq \mu(F \geq k), \qquad \Big(\int \Phi_k(F) d\mu\Big)^2 \leq \mu(F > k - 1)^2.$$

Almost surely one has $|\nabla\Phi_k(F)| \leq |\Phi_k'(F)| \cdot |\nabla F| \leq (1+3\varepsilon)L\mathbf{1}_{k-1<F<k}$. Therefore, letting ε to zero, the inequality

$$\mathbf{Var}(\Phi_k(F)) \leq \beta(s) \int |\nabla\Phi_k(F)|^2 d\mu + s\mathbf{Osc}(\Phi_k(F))^2,$$

readily implies

$$\mu(F > k) - \mu(F > k - 1)^2 \leq L^2\beta(s)\Big(\mu(F > k - 1) - \mu(F > k)\Big) + s.$$

Rearranging

$$\mu(F > k) \leq \frac{s}{1 + L^2\beta(s)} + \mu(F > k-1)\frac{\mu(F > k-1) + L^2\beta(s)}{1 + L^2\beta(s)}.$$

The first claimed inequality follows from the above and $\mu(F > k-1) \leq \mu(F > 0) \leq 1/2$. Iterating this inequality k times gives

$$\begin{aligned}
\mu(F > k) &\leq \frac{s}{1 + L^2\beta(s)}\left[1 + \left(1 - \frac{1}{2(1 + L^2\beta(s))}\right) + \cdots + \right. \\
&\quad \left. \left(1 - \frac{1}{2(1 + L^2\beta(s))}\right)^{k-1}\right] + \left(1 - \frac{1}{2(1 + L^2\beta(s))}\right)^k \mu(F > 0) \\
&\leq \frac{s}{1 + L^2\beta(s)} \cdot \frac{1}{1 - (1 - \frac{1}{2(1+L^2\beta(s))})} + \frac{1}{2}\left(1 - \frac{1}{2(1 + L^2\beta(s))}\right)^k \\
&\leq 2s + \frac{1}{2}\exp\left(\frac{-k}{2(1 + L^2\beta(s))}\right).
\end{aligned}$$

Note that this is also true when $k = 0$. Let $\lambda > 0$ and apply the latter bound to the λL-Lipschitz function λF with median 0. Denoting by $[x]$ the integer part of x, we get

$$\begin{aligned}
\mu(F > k) &= \mu(\lambda F > \lambda k) \leq \mu(\lambda F > [\lambda k]) \\
&\leq 2s + \frac{1}{2}\exp\Big(\frac{-[\lambda k]}{2(1 + \lambda^2 L^2\beta(s))}\Big) \\
&\leq 2s + \frac{1}{2}\exp\Big(\frac{-\lambda k}{2(1 + \lambda^2 L^2\beta(s))} + \frac{1}{2(1 + \lambda^2 L^2\beta(s))}\Big) \\
&\leq 2s + \frac{\sqrt{e}}{2}\exp\Big(\frac{-\lambda k}{2(1 + \lambda^2 L^2\beta(s))}\Big).
\end{aligned}$$

Choosing $\lambda = 1/(L\sqrt{\beta(s)})$ establishes (2). The rest of the statement easily follows. □

Next we give a few examples.

Example 7. If β has a finite limit at 0 then taking $s = 0$ in (2) recovers the well known exponential deviation inequality.

Example 8. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a 1-Lipschitz function with median m. We consider on $\mathbb{R}^n$ the n-fold product of $dm_\alpha(t) = \alpha(1 + |t|)^{-1-\alpha}/2\, dt$ denoted m_α^n. Since this measure satisfies a weak spectral gap inequality with $\beta(s) = c_\alpha(s/n)^{-2/\alpha}$, the deviations of $\mu(F - m > k)$ are controlled by

$$\inf_{s\in(0,1/4)} 2s + \frac{\sqrt{e}}{2}\exp\Big(\frac{-ks^{1/\alpha}}{4\sqrt{c_\alpha}n^{1/\alpha}}\Big).$$

Setting $t = k/(4\sqrt{c_\alpha}n^{1/\alpha})$, we choose $s = (\alpha\log(t)/t)^\alpha$. It is in the interval $(0, 1/4)$ provided t is larger than a constant $t_1(\alpha)$. Under this hypothesis the infimum is bounded from above by

$$2(\alpha \log(t)/t)^\alpha + 1/t^\alpha.$$

Therefore there exists constants $t_0(\alpha) > e$ and $C(\alpha)$ such that for $t \geq t_0(\alpha)$

$$m_\alpha^n(|F - m| > tn^{1/\alpha}) \leq C(\alpha) \left(\frac{\log(t)}{t}\right)^\alpha. \tag{3}$$

This is valid provided $4t\sqrt{c_\alpha} n^{1/\alpha} \in \mathbb{N}$ but extends to general values of t, with slightly worse constants. As we show next, this estimate is correct up to the log factor. Presumably, this point could be improved by optimizing in s the recursion formula (1).

Let us prove that (3) is very close to the truth, by adapting Talagrand's argument. It consists in analyzing product sets. First note that if $A \subset \mathbb{R}^n$ has measure at least $a \geq 1/2$ then 0 is a median of the distance function $x \mapsto d(x, A)$. Since the latter is 1-Lipschitz, (3) applies and gives,

$$m_\alpha^n(A_{tn^{1/\alpha}}) \geq 1 - C(\alpha) \left(\frac{\log t}{t}\right)^\alpha. \tag{4}$$

We show that this is close to optimal by choosing a specific product set. Namely we take $A = (-\infty, R^{-1}(a^{1/n})]^n$, where $R = R_{m_\alpha}$ is the distribution function of m_α and R^{-1} is its reciprocal function. By definition $m_\alpha^n(A) = a$. For $h > 0$, its h-enlargement satisfies

$$m_\alpha^n(A_h) \leq m_\alpha^n(A + [-h, h]^n) = m_\alpha^n((-\infty, R^{-1}(a^{1/n}) + h]^n) = R\Big(R^{-1}(a^{1/n}) + h\Big)^n.$$

The function R is explicitly computed. The latter estimate thus becomes

$$\begin{aligned} m_\alpha^n(A_h) &\leq \left(1 - \frac{1}{2\left(h + (2(1 - a^{1/n}))^{-1/\alpha}\right)^\alpha}\right)^n \\ &\leq \exp\left(\frac{-n}{2\left(h + (\frac{2}{n}\log(\frac{1}{a}) + O(\frac{1}{n^2}))^{-1/\alpha}\right)^\alpha}\right) \\ &= \exp\left(\frac{-1}{2\left(\frac{h}{n^{1/\alpha}} + (2\log(\frac{1}{a}) + O(\frac{1}{n}))^{-1/\alpha}\right)^\alpha}\right). \end{aligned}$$

We think of A and h as depending on n. The above bound shows that when n is large and $h << n^{1/\alpha}$ the measure of A_h is essentially equal to $a = m_\alpha^n(A)$. This confirms that $h = tn^{1/\alpha}$ is the right scale of enlargement. In this scale we have

$$m_\alpha^n(A_{tn^{1/\alpha}}) \leq \exp\left(\frac{-1}{2\left(t + (2\log(\frac{1}{a}) + O(\frac{1}{n}))^{-1/\alpha}\right)^\alpha}\right) \leq 1 - \frac{c_\alpha}{t^\alpha},$$

when $t \geq t_2(\alpha)$. Comparing this with Inequality (4) proves the tightness of our bounds.

Example 9. Finally, we consider the measures $\nu_p^n = \left(d_p e^{-|t|^p} dt\right)^{\otimes n}$, for $p \in (0,1)$. We have shown that they satisfy the weak Poincaré inequality with $\beta(s) = k_p \log(2n/s)^{(2/p)-2}$. Therefore the deviations of Lipschitz functions are controlled by

$$\inf_{s\in(0,1/4)} 2s + \exp\left(\frac{-k(\log(2n/s))^{1-1/p}}{4\sqrt{k_p}}\right).$$

We look for a value of s such that the two terms are of similar size. We are inspired by the case $p = 1/2$ where explicit calculations can be done.

If $k \geq (\log n)^{1/p}$ we set $s = 2e^{-k^p}$. The above infimum is at most (denoting by c_p a quantity depending only on p and that may be different in different occurrences)

$$\begin{aligned}
\nu_p^n(F - m > k) &\leq 4e^{-k^p} + e^{-kc_p(\log n + k^p)^{1-1/p}} \\
&\leq 4e^{-k^p} + e^{-kc_p(2k^p)^{1-1/p}} \\
&\leq 5e^{-c_p k^p}.
\end{aligned}$$

Here we did not check that the chosen s is less than $1/4$, since otherwise the bound is trivial.

If $k \leq (\log n)^{1/p}$ we set $s = 2e^{-k(\log n)^{1-1/p}}$. We get

$$\begin{aligned}
\nu_p^n(F - m > k) &\leq 4e^{-k(\log n)^{1-1/p}} + e^{-kc_p(\log n + k(\log n)^{1-1/p})^{1-1/p}} \\
&\leq 4e^{-k(\log n)^{1-1/p}} + e^{-kc_p(2\log n)^{1-1/p}} \\
&\leq 5e^{-c_p k(\log n)^{1-1/p}}.
\end{aligned}$$

As a conclusion we obtained

$$\nu_p^n(|F - m| > k) \leq 10 \exp\left(\frac{-c_p k}{\max(k^p, \log n)^{\frac{1}{p}-1}}\right).$$

In particular, for ε fixed and n large, it is enough to take $k \geq c_p(\log \frac{10}{\varepsilon})(\log n)^{\frac{1}{p}-1}$ in order to ensure $\nu_p^n(|F - m| > k) \leq \varepsilon$.

Remark 10. Theorem 2.4 of [16] also derives concentration inequalities from a weak spectral gap inequality, but they are different from ours. Comparing their Corollary 2.5 with the above examples shows that our result is sharper. The main technical reason for this is that the final step of our proof (which reintroduces homogeneity, as it was destroyed by the cut-off method) is not performed. Combining their method and the optimization on a scaling factor λ provides a slightly better estimate than ours. Let $c \in (0, 1/2)$, then with the notation of the theorem

$$\left(\frac{1}{2} - c\right)^2 \frac{k^2}{4} \leq \left[\log\left(\frac{1}{2\mu(F - m > k)}\right) + \frac{1}{2} - c\right] \int_{\mu(F-m>k)}^{\frac{1}{2}} \frac{\beta(cs)}{s} ds.$$

In general though, the integral can only be estimated by

$$\beta\big(c\mu(F-m>k)\big)\log\left(\frac{1}{2\mu(F-m>k)}\right).$$

This recovers our bound. For the measures m_α the integral can be computed and one gets a better decay, by a different power on the log-term. In the case of ν_p the explicit computation does not improve on our result.

6 Concave potentials of power type

In this section we apply our methods to products of probability measures on $\mathbb{R}$, $d\mu_\Phi(x) = Z_\Phi^{-1}e^{-\Phi(|x|)}dx$, where Φ satisfies the following assumption:

Hypothesis (H). *(i)* $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing concave function with $\Phi(0) = 0$ and $\mathcal{C}^2$ in a neighborhood of $+\infty$.

(ii) There exists $B > 1$ such that for x large enough $\Phi(2x) \geq B\Phi(x)$.

(iii) There exists $C > 0$ such that for x large enough $|x\Phi''(x)| \leq C\Phi'(x)$.

Hypothesis (H) naturally generalizes the power potentials $\Phi_p(x) = |x|^p$, $p \in (0,1)$. In particular it is not hard to check that $\Phi_{p,\beta} = |x|^p \log(\gamma + |x|)^\alpha$ with $p \in (0,1)$, $\alpha > 0$ and $\gamma = e^{2\alpha/(1-p)}$ verifies Hypothesis (H) with $B = 2^p$ and $C = 1$.

Remark 11. Assertion (ii) of (H) yields $\lim_{+\infty}\Phi = +\infty$ and by induction for large x

$$2\Phi(x) \leq \Phi(B'x), \tag{5}$$

with $B' = 2^{1+\log 2/\log B} > 1$. On the other hand, since Φ is concave and $\Phi(0) = 0$, (ii) also implies that

$$(B-1)\Phi(x) \leq \Phi(2x) - \Phi(x) \leq \int_x^{2x}\Phi' \leq x\Phi'(x) \leq \int_0^x \Phi' = \Phi(x) \tag{6}$$

where the left inequality is valid for x large enough, and the other ones for $x \geq 0$ (when Φ is not differentiable, $\Phi'(x)$ stands for the right derivative). Together with (iii) this result implies that $\dfrac{|\Phi''(x)|}{\Phi'(x)^2} \leq \dfrac{C}{x\Phi'(x)} \leq \dfrac{C}{(B-1)\Phi(x)}$. Hence, $\lim_{+\infty}\frac{|\Phi''|}{(\Phi')^2} = 0$. Also, combining the concavity assumption with (5) and (6) yields for x large enough

$$\Phi'(x) \geq \Phi'(B'x) \geq B''\Phi'(x), \tag{7}$$

where $B'' \in (0,1)$ depends only on B.

Now we prove that μ_Φ satisfies a weak Poincaré inequality with appropriate function β.

Proposition 9. *Let $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)}dx$ be a probability measure on $\mathbb{R}$. Assume that Φ verifies Hypothesis (H). Then there exists a constant $c_\Phi > 0$ such that μ_Φ satisfies a weak Poincaré inequality with function $c_\Phi\beta$ where*

$$\beta(s) = \frac{1}{[\Phi' \circ \Phi^{-1}(\log \frac{1}{s})]^2}, \quad s \in (0, 1/4).$$

Proof. We use Corollary 4. From Hypothesis (H) and the above remark there exists $A > 0$ such that, for $x > A$

$$\Phi'(x) \neq 0 \quad \text{and} \quad \frac{|\Phi''(x)|}{\Phi'(x)^2} \leq \frac{1}{2}.$$

Thus, we only have to check that $\beta\left(\frac{2e^{-\Phi(x)}}{|\Phi'(x)|}\right) \geq \frac{c}{\Phi'(x)^2}$ for some constant $c > 0$ and $|x|$ large enough.

It follows from Remark 11 that for x large $\frac{\log(\Phi'(x))}{\Phi(x)} \leq \frac{\log(x\Phi'(x))}{\Phi(x)} \leq \frac{\log(\Phi(x))}{\Phi(x)}$. Since $\lim_{+\infty} \Phi = +\infty$ we can deduce that $\lim_{+\infty} \frac{\log \Phi'}{\Phi} = 0$. Hence, for x large enough one has

$$\log \frac{1}{2} + \log \Phi'(x) + \Phi(x) \geq \frac{1}{2}\Phi(x).$$

Now Equation (5) implies that

$$\Phi^{-1}(\log \frac{1}{2} + \log \Phi'(x) + \Phi(x)) \geq \Phi^{-1}(\frac{1}{2}\Phi(x)) \geq \frac{x}{B'}.$$

Since Φ' is non-increasing the above inequality and (7) lead to

$$\begin{aligned} \beta\left(\frac{2e^{-\Phi(x)}}{|\Phi'(x)|}\right) &= \frac{1}{(\Phi')^2 \circ \Phi^{-1}(\log \frac{1}{2} + \log \Phi'(x) + \Phi(x))} \\ &\geq \frac{1}{(\Phi')^2(x/B')} \geq \left(\frac{B''}{\Phi'(x)}\right)^2 \end{aligned}$$

for x large enough. This achieves the proof. □

Example 12. This result recovers the case $\Phi_p = |x|^p$, $p \in (0,1)$. For $\Phi_{p,\alpha} = |x|^p \log(\gamma + |x|)^\alpha$ with $p \in (0,1)$, $\alpha > 0$ and $\gamma = e^{2\alpha/(1-p)}$, one can easily see that $\mu_{p,\alpha}$ satisfies a weak Poincaré inequality with function asymptotically (when s is small) behaving like

$$\beta_{p,\alpha}(s) = \frac{1}{(\log \frac{1}{s})^{2(1-\frac{1}{p})} (\log \log \frac{1}{s})^{\frac{2\alpha}{p}}}.$$

We obtain the following concentration inequalities for μ_Φ^n:

Proposition 10. *Let $d\mu_\Phi(x) = Z_\Phi^{-1} e^{-\Phi(|x|)}dx$ be a probability measure on $\mathbb{R}$ which verifies Hypothesis (H). Then there exist $c_\Phi, \tilde{c}_\Phi, k_\Phi > 0$ such that for any $n \geq 1$, any 1-Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$ and any integer $k \geq k_\Phi$ one has*

$$\begin{aligned} \mu_\Phi^n(|F - m| > k) &\leq 6 \exp\left(-c_\Phi k \Phi' \circ \Phi^{-1}(\max(\Phi(k), 2\log n))\right) \\ &\leq 6 \max\left(e^{-\tilde{c}_\Phi \Phi(k)}, e^{-c_\Phi k \Phi' \circ \Phi^{-1}(2\log n)}\right). \end{aligned}$$

where m is a median of F.

Proof. As in the previous section, since μ^n satisfies a weak Poincaré inequality with function $\beta(s) = c_\Phi/[\Phi' \circ \Phi^{-1}(\log \frac{n}{s})]^2$, the deviations of 1-Lipschitz functions are controlled by

$$\inf_{s\in(0,1/4)} 2s + \frac{\sqrt{e}}{2}\exp\left(-\frac{4}{\sqrt{c_\Phi}}k\Phi' \circ \Phi^{-1}(\log \frac{n}{s})\right).$$

Next we look for a value of s such that the two terms are of similar size. We will denote by c_Φ a quantity depending only on Φ that may change from line to line. We work with k large enough in order to be able to use the doubling condition in the following arguments.

If $k \geq \Phi^{-1}(\log n)$ we set $s = e^{-\Phi(k)}$. The above infimum is at most

$$\begin{aligned}
\mu_\Phi^n(F - m > k) &\leq 2e^{-\Phi(k)} + e^{-c_\Phi k\Phi'\circ\Phi^{-1}(\log n+\Phi(k))} \\
&\leq 2e^{-\Phi(k)} + e^{-c_\Phi k\Phi'(k)} \\
&\leq 3e^{-c_\Phi k\Phi'(k)} \leq 3e^{-\tilde{c}_\Phi \Phi(k)}.
\end{aligned}$$

Here, we have used Equation (5) in order to get that

$$\Phi^{-1}(\log n + \Phi(k)) \leq \Phi^{-1}(2\Phi(k)) \leq B'k,$$

and thus by (7), $\Phi' \circ \Phi^{-1}(\log n + \Phi(k)) \geq B''\Phi'(k)$. The last inequality comes from (6).

If $k < \Phi^{-1}(\log n)$ we set $s = e^{-k\Phi'\circ\Phi^{-1}(\log n)}$. Recall first that Inequality (6) asserts that for $x \geq 0$ one has $x\Phi'(x) \leq \Phi(x)$. Hence $\Phi^{-1}(x)\Phi' \circ \Phi^{-1}(x) \leq x$ and in turn it follows that

$$k\Phi' \circ \Phi^{-1}(\log n) \leq \Phi^{-1}(\log n)\Phi' \circ \Phi^{-1}(\log n) \leq \log n.$$

We get

$$\begin{aligned}
\mu_\Phi^n(F - m > k) &\leq 2e^{-k\Phi'\circ\Phi^{-1}(\log n)} + e^{-c_\Phi k\Phi'\circ\Phi^{-1}(\log n+k\Phi'\circ\Phi^{-1}(\log n))} \\
&\leq 2e^{-k\Phi'\circ\Phi^{-1}(\log n)} + e^{-c_\Phi k\Phi'\circ\Phi^{-1}(2\log n)} \\
&\leq 3e^{-c_\Phi k\Phi'\circ\Phi^{-1}(2\log n)}.
\end{aligned}$$

The result easily follows. □

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